FUZZY DIFFERENTIAL INCLUSIONS
AND NONPROBABILISTIC LIKELIHOOD

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ABSTRACT. By interpreting the right-hand side of a differential inclusion as a fuzzy set, we define a general concept of likelihood for its solutions, in a nonprobabilistic setting. We prove that under natural conditions there exists a solution having maximal likelihood. Given a convex valued, Lipschitzean differential inclusion $x' \in F(x)$, we propose two particular definitions of likelihood, relying on the distance between the derivative of a solution x and the (relative) boundary of $F(x)$. We use them to characterize the points which lie on the boundary of the funnel as those of minimal likelihood, and to derive other qualitative properties. We also study the multivalued selection from F consisting of the points in $F(x)$ having maximal distance from the boundary of $F(x)$.

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1. INTRODUCTION

Dynamical systems whose evolution law has some uncertainty may be governed by differential inclusions: the system can adopt any trajectory $x(\cdot)$ such that $x'(t)$ belongs to the set of feasible velocities corresponding to the state $x(t)$, namely

$$x'(t) \in F(x(t)), \qquad t \in [0, T] . \qquad (1.1)$$

If some additional informations are available, the modeler may define a rule to select from the set of solutions those which are supposed to be more fit to describe the system; in other words, to choose the "most likely" trajectories. For example, in economics "slow" or "heavy" solutions were introduced, see [3], [6], while in population biology solutions having "maximal growth" were defined, see [13]. In the case of absence of other informations, an intrinsic way

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of choosing "the most likely path" of a differential inclusion was proposed by
A. Bressan [8] and A. Cellina and R.M. Colombo [10]. This method, entirely
independent of probabilistic considerations, was called "metric", because it
relies on the metric structure of a space of functions. It provides a map on
the set of trajectories of (1.1) which is not a measure, being not additive: a
number $\mathcal{L}_\beta(x)$, called "likelihood", is assigned to a solution x of (1.1), given
by the infimum of the (Hausdorff or Kuratowski) measure of noncompactness
in an L_p space of the sets

$$\{y' \mid y' \in F(y),\ \|y(t) - x(t)\| < \varepsilon,\quad \forall t \in [0,T]\} \tag{1.2}$$

as $\varepsilon \to 0$. According to this definition, a solution of (1.1) is more likely than
another if there are more solutions of (1.1) (in the sense of the measure of
noncompactness) close to it.

In this paper we try to continue the qualitative analysis of differential
inclusions on the same line of the above considerations. We remark that
the concept of fuzzy sets and of fuzzy differential inclusions seems to be
suitable to associate a likelihood with solutions of (1.1) in a nonprobabilistic
framework. A fuzzy set F can be represented by a "membership function"
$\mathcal{F}(x)$, which to every point x of a space X assigns a value $\mathcal{F}(x)$ indicating
how much x is "in"F. We will consider membership functions having values
in $\{-\infty\} \cup [0,1]$, the best value being 1. By using fuzzy sets, J.P. Aubin
[3] introduced the concept of fuzzy differential inclusions as problems of the
form (up to the sign of membership functions)

$$\mathcal{F}(x(t), x'(t)) > -\infty\ ,\qquad t \in [0,T]\ ; \tag{1.3}$$

here $\mathcal{F}$ denotes a fuzzy set-valued map, i.e. a map that associates with every
point $x \in X$ a fuzzy set (represented through its membership function)
$\mathcal{F}(x, \cdot) : X \to \{-\infty\} \cup [0,1]$. Having the fuzzy differential inclusion (1.3), it
seems natural to consider the solution set S_F of (1.3) as a fuzzy set with the
membership function

$$\mathcal{L}(v) := 1/T \int_0^T \mathcal{F}(v(t), v'(t))dt.$$

For $v \in S_F$ the value $\mathcal{L}(v)$ may be considered as a measure of the likelihood
of the solution v. This concept may be useful as an alternative to the use
of stochastic differential equations (see [1]). A study of fuzzy differential
inclusions may be then motivated as an appropriate environment to define a
nonprobabilistic likelihood. We also define the likelihood of a point ξ in the
reachable set $R_F(T)$ of (1.3) at the time T by

$$L(\xi) := \sup\{\mathcal{L}(u) \mid u \in S_F, u(T) = \xi\}. \tag{1.4}$$

The above definition generalizes the concept of metric likelihood given in [8] and [10]. In fact the main result of those papers is a representation of $\mathcal{L}_\beta(\cdot)$ as an integral functional, whose integrand may be viewed as a particular membership function for $F(x)$. Roughly speaking, this function depends on the distance between the derivative $x'(t)$ of a solution of (1.1) from the set of extreme points of $F(x(t))$, assigning minimal likelihood to the solutions of $x' \in \text{ext}F(x)$. We recall that the idea of calculating the measure of noncompactness of a class of sets of functions through an integral functional goes back to [11].

In the first part of this paper (§3) we sum up some basic properties concerning the fuzzy likelihood, e.g. the existence of solutions of (1.3) with maximal likelihood. We also introduce the concept of solutions with maximal "myopic" likelihood, i.e. solutions satisfying

$$\sup_{u \in X} \mathcal{F}(v(t), u) = \mathcal{F}(v(t), v'(t)) > -\infty , \tag{1.5}$$

and show that there exist solutions satisfying (1.5). This approach generalizes the concept of γ-maximal solutions, introduced in [16] for differential inclusions on closed sets.

In the second part of this paper (§4,5) we associate two particular membership functions with the differential inclusion (1.1) and investigate some of their properties. Roughly speaking, the first function, called $\mathcal{F}_\partial$, measures the distance of a point $y \in F(x)$ from the boundary of the set $F(x)$, while the second one, called $\mathcal{F}_b$, the distance of $y \in F(x)$ from the relative boundary of $F(x)$. Our choice assigns the minimal likelihood $\mathcal{L}_\partial$ (resp. $\mathcal{L}_b$) to all the solutions of $x'(t) \in \text{bd}F(x(t))$ (resp. of $x'(t) \in \text{rel bd}F(x(t))$). With our definitions the set of solutions of (1.1) and of (1.3) coincide. Section 5 contains some qualitative results on the reachable sets of Lipschitzean differential inclusions, obtained using $\mathcal{L}_\partial$. In particular, if F has compact and convex values with nonempty interior, we characterize the boundary of the funnel of (1.1) from a fixed initial value as the set of points in it having minimal likelihood. This property may be viewed as parallel to Hukuhara's theorem for differential inclusions (see Theorem 2.2.4 in [5]): from $\xi \in \text{bd}R_F(T)$ we deduce that every solution of (1.1) reaching it is such that $x'(t) \in \text{bd}F(x(t))$ almost everywhere. We also give a refinement of Hukuhara's theorem itself. Moreover, we show that the map L_∂, defined on the reachable set as in (1.4), is continuous. We recall that the definition in [8] and [10] provides a map which is upper, but not lower, semicontinuous (see [10], Example 3.7).

Finally, §6 contains an analysis of the multivalued selection $C_F(x)$ from F consisting of those points in $F(x)$ having maximal distance from $\text{bd}F(x)$. It is shown that it is upper semicontinuous (in finite dimensions) and it may not be Lipschitzean if F is so. This investigation is motivated by the

Then we can choose a weakly convergent subsequence (in $L_1[0,T]$) $v'_{n_k}(\cdot) \to v'(\cdot)$. Since $\mathcal{F}$ is concave in the second variable, from Theorem 10.8.i, p. 352 in [12] it follows

$$\int_0^T \mathcal{F}(v(t),v'(t))dt \geq \limsup_{k\to\infty} \int_0^T \mathcal{F}(v_{n_k}(t),v'_{n_k}(t))dt \geq \lambda.$$

$\triangle$

Let us recall that, given the fuzzy differential inclusion (1.3), two concepts of solutions having maximal likelihood may be considered. The first one is based on the so-called "intertemporal" optimization, i.e. one looks for a trajectory maximizing the integral functional $\mathcal{L}(v)$ on the set of solutions of (1.3). This concept assumes an insight into the future. The other approach, called "myopic" optimization, looks for a solution of (1.3) such that the equality (1.5) holds.

Remark 3.1. The existence of a solution with maximal intertemporal likelihood $\mathcal{L}$ is a straightforward consequence of Proposition 3.1.

The next proposition ensures the existence of a solution having maximal myopic likelihood.

Proposition 3.2. *Let* $K := B[0,MT]$. *Let* $\mathcal{F} : K \times X \to [0,1] \cup \{-\infty\}$ *be a locally bounded fuzzy set-valued map, continuous on its domain, with convex non-trivial values. Then there exists a solution to (1.5).*
Proof. Let us consider the set-valued map

$$G(x) := \{y \in X \mid \mathcal{F}(x,y) = \sup_{z\in X} \mathcal{F}(x,z)\}.$$

Then $G(B[0,MT]) \subset B[0,M]$, and from the marginal map theorem in [6] it follows that the map G has closed graph and convex compact values. Therefore there exists a solution v to $x'(t) \in G(x(t))$. This solution satisfies (1.5).
$\triangle$

Remark 3.2. We remark that the myopic maximization of likelihood generalizes the problem of finding γ-maximal solutions, i.e. solutions to the system

$$\begin{cases} x'(t)) \in G(x(t)), \\ \gamma(x'(t)) = \sup_{z\in G(x(t))} \gamma(z), \\ x(0) = 0, \end{cases}$$

where $\gamma : X \mapsto I\!R$ is a given function. For differential inclusions on closed sets the existence of γ-maximal solutions was studied in [16]. In fact, such

a solution can be viewed as a trajectory with maximal myopic likelihood for the fuzzy differential inclusion (1.3), if we define

$$\mathcal{F}(x,y) := \begin{cases} -\infty & \text{if } y \notin G(x) \\ \gamma(y) & \text{if } y \in G(x). \end{cases}$$

$\triangle$

Finally, the reachable set $R_F(T)$ may be regarded as a fuzzy set with the following membership function:

Definition 3.4. *For every $\xi \in X$ we define*

$$L(\xi) := \begin{cases} -\infty & \text{if } \xi \notin R_F(T) \\ \sup\{\mathcal{L}(u) \mid u \in \mathcal{S}_F, u(T) = \xi\} & \text{otherwise.} \end{cases}$$

Remark 3.3. Being $\mathcal{L}$ upper semicontinuous and $\mathcal{S}_F$ compact, the supremum in Definition 3.4 is attained. Moreover, the map $L : R_F(T) \to \mathbb{R}^+$ is upper semicontinuous. In fact, let $(\xi_n)_{n \geq 1}$ be a sequence in $R_F(T)$ converging to ξ and let $(x_n)_{n \geq 1}$ be solutions of (5.1), (5.2) such that $L(\xi_n) = \mathcal{L}(x_n)$. By compactness there exists a subsequence x_{n_k} uniformly converging to x. By the upper semicontinuity of $\mathcal{L}$,

$$L(\xi) \geq \mathcal{L}(x) \geq \limsup_{k \to \infty} \mathcal{L}(x_{n_k}) = \limsup_{k \to \infty} L(\xi_{n_k}).$$

Since the above argument can be repeated for every subsequence of (ξ_n), the upper semicontinuity of L is proved.

$\triangle$

4. TWO METRIC LIKELIHOODS

Let $F : B[0, TM] \to 2^X$ be a continuous set-valued map with convex and compact values, such that $F(B[0, TM]) \subset B[0, M]$, and consider the initial value problem

$$\begin{cases} x'(t) \in F(x(t)), \\ x(0) = 0. \end{cases} \tag{4.1}$$

In this section we introduce two membership functions for the map F and define the corresponding likelihood functions for the solution set $\mathcal{S}_F$ and the reachable set $R_F(T)$ of (4.1). We call them "metric" likelihoods since they are essentially based on the Euclidian distance in X.

Definition 4.1. *Let $F : X \to 2^X$ be a set-valued map with values having nonempty interior. We define for it the fuzzy set-valued map*

$$\mathcal{F}_0(x,y) := \begin{cases} d(y, \mathrm{bd}\, F(x)) / \sup_{z \in F(x)} d(z, \mathrm{bd}\, F(x)) & \text{if } y \in F(x) \\ -\infty & \text{otherwise.} \end{cases}$$

For each $v \in S_F$ we define its ∂-likelihood on the interval $[0,T]$ as

$$\mathcal{L}_\partial(v) := 1/T \int_0^T \mathcal{F}_\partial(v(t), v'(t))dt. \qquad (4.2)$$

Remark 4.1. By Lemma 6.1 below and Proposition 2.1 and Lemma 3.1 in [15] it follows that, for every $x \in B[0, TM]$, $\mathcal{F}_\partial(x, \cdot)$ is concave, and $\mathcal{F}_\partial(\cdot, \cdot)$ is upper semicontinuous and continuous on its domain. Hence (4.2) defines a fuzzy likelihood that satisfies the assumptions of the Proposition 3.1 and 3.2.

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Our second definition provides a notion of likelihood which is meaningful also for set-valued maps with values having everywhere empty interior. Denote by $b(x)$ the barycenter of $F(x) + B$ (see §1.9 in [5]), i.e.

$$b(x) := \frac{1}{m_n(F(x) + B)} \int_{F(x)+B} y\, dm_n,$$

which is well defined because $F(x) + B$ has positive measure in $X = \mathbb{R}^n$. By Lemma 9.2 in [9], $b(x) \in \text{ri} F(x)$ and, by Theorem 1, p.77 in [5], $b(\cdot)$ is a continuous selection from $F(\cdot)$. If moreover $F(\cdot)$ is Lipschitzean then $b(\cdot)$ is also Lipschitzean.

Definition 4.2. *Let $F : X \to 2^X$ be a set-valued map. We define for it the fuzzy set-valued map $\mathcal{F}_b$ as*

$$\mathcal{F}_b(x, y) := \begin{cases} \sup\{z \in \mathbb{R} \mid (y, z) \in \text{co}\{(F(x) \times \{0\}) \cup \{(b(x), 1)\}\}\} & \text{if } y \in F(x) \\ -\infty & \text{otherwise.} \end{cases}$$

For each $v \in S_F$ we define its b-likelihood on the interval $[0, T]$ as

$$\mathcal{L}_b(v) := 1/T \int_0^T \mathcal{F}_b(v(t), v'(t))dt. \qquad (4.3)$$

Remark 4.2. $\mathcal{F}_b$ is by construction concave in the second variable; it is also jointly upper semicontinuous and continuous on its domain, by Lemma 16 in §5 of [17]. Therefore (4.3) defines a fuzzy likelihood that satisfies the assumptions of Proposition 3.1 and 3.2.

$\triangle$

The two preceding definitions follow the same line: the first one assigns zero likelihood to solutions satisfying

$$x'(t) \in \text{bd} F(x(t)) \qquad \text{for a.e. } t \in [0, T],$$

while the second one to solutions for which

$$x'(t) \in \mathrm{rb}F(x(t)) \qquad \text{for a.e. } t \in [0, T].$$

The trajectories of (4.1) which maximize the ∂-likelihood in the myopic sense are the solutions of the differential inclusion

$$\begin{cases} x'(t) \in C_F(x(t)), \\ x(0) = 0, \end{cases} \tag{4.4}$$

where

$$C_F(x) = \{y \in F(x) \mid d(y, \mathrm{bd}F(x)) = \sup_{z \in F(x)} d(z, \mathrm{bd}F(x))\}.$$

As it is shown in the Example 6.2 b) below, the problem (4.4) may not have a unique solution, even if F is Lipschitzean. On the other hand, the solutions of (4.1) which maximize the b-likelihood in the myopic sense are those satisfying

$$\begin{cases} x'(t) = b(x(t)), \\ x(0) = 0. \end{cases}$$

Therefore, if F is Lipschitzean, there exists a unique solution to (4.1) having maximal b-likelihood (in both myopic and intertemporal sense).

Remark 4.3. Let $\mathcal{L}_\beta$ be the metric likelihood defined in [8] and [10],

$$\mathcal{L}_\beta(v) = \lim_{\epsilon \to 0} \beta(\{u' \mid u \in \mathcal{S}_F, \|u - v\| < \epsilon\}).$$

By the representation theorem 3.2 in [8], if F is Lipschitzean, compact and convex valued, the β-likelihood of a solution v of (4.1) is given by

$$\mathcal{L}_\beta(v) = \left(\int_0^T h^2(v'(t), F(v(t)))\, dt\right)^{1/2},$$

where for a nonempty compact convex subset Ω of X and $\omega \in \Omega$ the function h is defined as

$$h(\omega, \Omega) = \sup\left\{\left(\int_0^1 |f(\xi) - \omega|^2\, d\xi\right)^{1/2} \mid f : [0,1] \to \Omega, \int_0^1 f(\xi)\,d\xi = \omega\right\},$$

and $h(\omega, \Omega) = -\infty$ if $\omega \notin \Omega$. The function h is also proved to be upper semicontinuous and concave in ω. The analogous Theorem 4.5 in [10] constructs

for a map F of the form $r(t) + s(t)\mathrm{bd}U$, U convex (F independent of x), another function Δ with similar properties, and represents $\mathcal{L}_\beta(v)$ as

$$\left(\int_0^T [\Delta(v'(t))]^p \, dt \right)^{1/p} .$$

Hence, the definition of fuzzy likelihood of the preceding paragraph contains the definition of $\mathcal{L}_\beta$, provided the L_1 norm is substituted by the L_p one. As a consequence of both representations, $\mathcal{L}_\beta(v) = 0$ implies

$$x'(t) \in \mathrm{ext}F(x(t)) \qquad \text{for a.e. } t \in [0, T],$$

hence both $\mathcal{L}_\vartheta(v)$, $\mathcal{L}_b(v)$ are zero.

5. ON THE FUNNEL OF LIPSCHITZEAN DIFFERENTIAL INCLUSIONS

Let $M > 1$ and $F : B[0, MT] \to 2^X$ be a Lipschitzean multifunction with compact and convex values, such that $F(B[0, MT]) \subset B[0, M]$. We consider the set of solutions of

$$\begin{cases} x'(t) \in F(x(t)), \\ x(0) = 0, \end{cases} \tag{5.1}$$

and study the likelihood $\mathcal{L}_\vartheta$, defined in the previous paragraph for $x \in \mathcal{S}_F$, and the likelihood of a point $\xi \in R_F(\tau)$, $0 < \tau < T$,

$$L_\vartheta(\xi) = \max\{\mathcal{L}_\vartheta(x) \,|\, x \in \mathcal{S}_F, x(\tau) = \xi\},$$

where we consider $\mathcal{L}_\vartheta$ as an integral over the interval $[0, \tau]$. Here the normalization of $\mathcal{F}_\vartheta$ can be ignored, in order to allow $F(x)$ to have empty interior for some x: we set, for $x \in \mathcal{S}_F$, $\mathcal{L}_\vartheta(x) = \int_0^T d(x(t), \mathrm{bd}F(x(t))) \, dt$. The main technical tool of this paragraph is the fact that a solution x of (5.1) is also the solution of a Lipschitzean differential equation "within the differential inclusion", as it was pointed out by several authors (see [21], [19], [2]). Namely, there exists a Carathéodory Lipschitzean selection $f(t, x)$ from $F(x)$ such that $x'(t) = f(t, x(t))$ for almost every $t \in [0, T]$.

The following properties hold.

Theorem 5.1. *Let $0 < \tau \leq T$ and ξ be in $R_F(\tau)$. Then $(\tau, \xi) \in \mathrm{bd}\tilde{R}_F$ implies $L_\vartheta(\xi) = 0$ on the interval $[0, \tau]$. As a partial converse, if $0 \in \mathrm{int}F(0)$ or if $\mathrm{int}F(\xi) \neq \emptyset$, then $L_\vartheta(\xi) = 0$ on the interval $[0, \tau]$ implies $(\tau, \xi) \in \mathrm{bd}\tilde{R}_F$.*

Proof. Assume, by contradiction, that there exist $(\tau, \xi) \in \text{bd} \tilde{R}_F$, $x \in S_F$ and $E \subset [0, \tau]$ such that $\xi = x(\tau)$, but $\text{m}(E) > 0$ and $x'(t) \in \text{int} F(x(t))$ $\forall t \in E$. By Theorem 1 in [21] there exists a Carathéodory function $f : [0, T] \times B[0, MT] \to X$, Lipschitzean in x for a.e. t, such that $f(t, x) \in F(x)$ for all t, x, and $x'(t) = f(t, x(t))$ for almost every $t \in [0, T]$. By possibly taking a subset, we can assume that $f(\cdot, \cdot)$ is jointly continuous on $E \times B[0, MT]$ (Scorza Dragoni theorem). Let $\tau > t_o \in E$ be a density point of E such that $x'(t_o) = f(t_o, x(t_o))$. By the Hausdorff continuity of F, there exist $r > 0$ and $\eta > 0$ such that

$$f(t_o, x(t_o)) + rB \subset F(y) \qquad \text{if } |y - x(t_o)| < \eta . \qquad (5.2)$$

Choose t_1, $\tau > t_1 > t_o$, such that

$$t_1 - t_o < \frac{\eta}{M}, \qquad (5.3)$$

$$\text{m}(E \cap [t_o, t_1]) \geq (t_1 - t_o)(1 - \frac{r}{6M}), \qquad (5.4)$$

$$|f(t_o, x(t_o)) - f(t, x(t))| < \frac{r}{3} \qquad \forall t \in E \cap [t_o, t_1], \qquad (5.5)$$

and define

$$G(t, y) = \begin{cases} f(t, y) & \text{if } t \notin [t_o, t_1], \\ f(t_o, x(t_o)) + rB & \text{if } t \in [t_o, t_1]. \end{cases}$$

Then $G(t, y)$ is measurable in t and Lipschitzean in y, and $G(t, y) \subset F(y)$ if $t \notin [t_o, t_1]$ or if $t \in [t_o, t_1]$ and $|y - x(t_o)| < \eta$. In particular, by (5.2) and (5.3) all the solutions of

$$\begin{cases} u'(t) \in G(t, u(t)) \\ u(0) = 0 \end{cases} \qquad (5.6)$$

are also solutions of (5.1). Thus, $\tilde{R}_G \subset \tilde{R}_F$.

We claim that $(\tau, \xi) \in \text{int} \tilde{R}_G$, reaching a contradiction with $(\tau, \xi) \in \text{bd} \tilde{R}_F$. Indeed, $R_G(t_1) = x(t_o) + (t_1 - t_o) f(t_o, x(t_o)) + (t_1 - t_o)rB$. Moreover, if we set z to be $x(t_o) + (t_1 - t_o) f(t_o, x(t_o))$, then, by (5.4) and (5.5)

$$\begin{aligned} |x(t_1) - z| &\leq \int_{t_o}^{t_1} |x'(t) - f(t_o, x(t_o))| \, dt \\ &\leq \int_{[t_o, t_1] \cap E} |f(t, x(t)) - f(t_o, x(t_o))| \, dt + 2 \int_{[t_o, t_1] \setminus E} M \, dt \\ &\leq \frac{r}{3}(t_1 - t_o) + \frac{r}{3}(t_1 - t_o) < r(t_1 - t_o), \end{aligned}$$

which implies that $x(t_1) \in \text{int} R_G(t_1)$. For $t > t_1$, $G(t,x)$ and $f(t,x)$ coincide. Since $f(t,\cdot)$ is Lipschitzean, for all $t > t_1$ the point $(t,x(t))$ belongs to the interior of the funnel of $u' = f(t,u)$, $u(t_1) \in R_G(t_1)$.

To prove the second part of the statement, assume by contradiction that $(\tau,\xi) \in \text{int} \tilde{R}_F$ and let $0 < \tau_1 < \tau$ and $r > 0$ be such that $B[\xi,r] \subset R_F(\sigma)$ for all $\tau_1 \leq \sigma \leq \tau$. If $0 \in \text{int} F(0)$, choose $x \in S_F$ such that $\xi = x(\tau_1)$ and set

$$y(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau - \tau_1 \\ x(t-(\tau-\tau_1)) & \text{if } t > \tau - \tau_1; \end{cases}$$

then $y(\tau) = \xi$ and $\mathcal{L}_\partial(y) > 0$ on $[0,T]$. If $\text{int} F(\xi) \neq \emptyset$, we can assume that there exists $v \in \text{int} F(\xi_1)$ for every $\xi_1 \in B[\xi,r]$. Let $\tau_1 < \tau' < \tau$ be such that $(\tau - \tau')\|v\| < r$ and choose $x \in S_F$ such that $x(\tau') = \xi + (\tau' - \tau)v$. Set

$$y(t) = \begin{cases} x(t) & \text{if } 0 \leq t \leq \tau' \\ \xi + (t-\tau)v & \text{if } \tau' \leq t \leq \tau; \end{cases}$$

then $y(\tau) = \xi$ and $\mathcal{L}_\partial(y) > 0$ on $[0,\tau]$, concluding the proof.

$\triangle$

Remark 5.1. If all the values of F have empty interior, the second part of Theorem 5.1 is no longer true, because it may happen that $\text{int} \tilde{R}_F \neq \emptyset$. Indeed, let $G : I\!\!R \to I\!\!R^2$ be defined by

$$G(r) = \begin{cases} \{(v,0) \mid -1 \leq v \leq +1\} & \text{if } r \leq 1/2 \\ \{(v\cos 2\pi(r-1/2), v\sin 2\pi(r-1/2)) \mid -1 \leq v \leq +1\} & \text{if } r \geq 1/2, \end{cases}$$

and set $F : I\!\!R^3 \to 2^{I\!\!R^3}$ to be

$$F(x_1,x_2,x_3) = \{(y_1,y_2,y_3) \mid 2 \leq y_1 \leq 3, \ (y_2,y_3) \in G(x_1)\} .$$

Clearly, $F(\cdot,\cdot,\cdot)$ is Lipschitzean with convex, compact values with empty interior. However,

$$\tilde{R}_F \supset \left\{ (t,t\xi_1,\xi_2,\xi_3) \mid t > 1/4, \ 2 \leq \xi_1 \leq 3, \ (\xi_2,\xi_3) \in \int_0^t G(\xi_1 s)\, ds \right\}$$

$$= \left\{ (t,t\xi_1,\xi_2,\xi_3) \mid t > 1/4, \ 2 \leq \xi_1 \leq 3, \right.$$

$$\frac{-\sin 2\pi(\xi_1 t - 1/2)}{2\pi\xi_1} \leq \xi_2 \leq \frac{+\sin 2\pi(\xi_1 t - 1/2)}{2\pi\xi_1},$$

$$\left. \frac{\cos 2\pi(\xi_1 t - 1/2) - 1}{2\pi\xi_1} \leq \xi_3 \leq \frac{1 - \cos 2\pi(\xi_1 t - 1/2)}{2\pi\xi_1} \right\} ,$$

which has nonempty interior.

<div align="right">△</div>

The technique of the parametrization of the right-hand side of (5.1) can also be used to prove two properties which provide an analogy between Lipschitzean differential equations and inclusions: we show that the multivalued flow generated by F is open, and that a solution x of (5.1) is a boundary solution on an interval $[0, \tau]$ (in the sense that $x(t) \in \mathrm{bd}R_F(t)$ for all $0 \le t \le \tau$) if and only if $x(\tau) \in \mathrm{bd}R_F(\tau)$. This statement refines the multivalued version of Hukuhara's theorem (see Theorem 2.2.4 in [5]) in the case of Lipschitzean differential inclusions.

Proposition 5.1. *Let $0 < \tau \le T$. Then*

(i) the multivalued flow $\xi \mapsto R_F^{(0,\xi)}(\tau)$ generated in X by the map F is open, in the sense that the set $\bigcup_{\xi \in \Omega} R_F^{(0,\xi)}(\tau)$ is open if $\Omega \subset B[0, MT]$ is open;

hence

(ii) if $\xi \in \mathrm{bd}R_F(\tau)$, for every $x(\cdot) \in S_F$ such that $x(\tau) = \xi$ we have that $x(t) \in \mathrm{bd}R_F(t)$ for all $t \in [0, \tau]$.

Proof. (i) Let $\xi' \in R = \bigcup_{\xi \in \Omega} R_F^{(0,\xi)}(\tau)$ and let $x \in S_F$ be such that $x(0) \in \Omega$ and $x(\tau) = \xi'$. Let, by Theorem 1 in [21], $f : [0, T] \times B[0, MT] \to X$ be a Carathéodory (single-valued) function, Lipschitzean in x for a.e. t, such that $f(t, x) \in F(x)$ for all t, x, and $x'(t) = f(t, x(t))$ for a.e. $t \in [0, T]$. Since f is Lipschitzean in x, the set $\bigcup_{\xi \in \Omega} R_f^{(0,\xi)}(\tau) \subset R$ contains ξ' in its interior.

(ii) Let $x \in S_F$ be such that $x(\tau) = \xi \in \mathrm{bd}R_F(T)$, but $x(t_o) \in \mathrm{int}R_F(t_o)$ for some $0 < t_o < \tau$. Let $r > 0$ be such that $B[x(t_o), r] \subset R_F(t_o)$. By (i), $x(\tau) \in \mathrm{int} \bigcup_{|\xi - x(t_o)| < r} R_F^{(t_o, \xi)}(\tau) \subset R_F(\tau)$, a contradiction.

<div align="right">△</div>

Remark 5.2. In [18] the author proves the following property:

If $x(\cdot)$ is a boundary trajectory of $x' \in F(t, x)$, $x(t_o) = x_o$ on $[t_o, T]$, i.e. $x(t) \in \mathrm{bd}\tilde{R}_F$ for all $t \in]t_o, T[$, then $x'(t) \in \mathrm{bd}F(t, x(t))$ for a.e. $t \in [t_o, T]$.

In that paper F is assumed only to be a (convex-valued) Carathéodory map. By Proposition 5.1, the first part of Theorem 5.1 can be obtained also from the above property. We mention finally that [20] contains a result stronger than Theorem 5.1, in the case where $\mathrm{bd}F(x)$ is a $\mathcal{C}^1$-smooth hypersurface of $I\!\!R^n$: the authors prove that for a boundary trajectory of (5.1) the Bouligand contingent cone to $F(x(t))$ at $x'(t)$ and to $R_F(t)$ at $x(t)$ coincide a.e.

<div align="right">△</div>

Theorem 5.2. *The map $\xi \mapsto L_\theta(\xi)$ from $R_F(T)$ into $I\!\!R^+$ is continuous.*

Proof. By the Remarks 4.1 and 3.3 above, $L_\theta(\cdot)$ is upper semicontinuous.

To show the lower semicontinuity, we recall first that $L_\theta(\cdot) \geq 0$, so that L_θ is automatically lower semicontinuous at every ξ such that $L_\theta(\xi) = 0$. Let $\overline{\xi} \in R_F(T)$ be such that $L_\theta(\overline{\xi}) > 0$ and let $\varepsilon > 0$ be given. We want to show that there exists $\delta > 0$ such that $|\xi - \overline{\xi}| < \delta$ implies $L_\theta(\xi) - L_\theta(\overline{\xi}) \geq -\varepsilon$. Choose, by the Remarks 4.1 and 3.3, $x(\cdot) \in S_F$ such that $\mathcal{L}_\theta(x) = L_\theta(\overline{\xi})$ and let $E \subset [0, T]$ be such that $m(E) > 0$ and $x'(t) \in \text{int} F(x(t))$ for all $t \in E$. Let, by Theorem 1 in [21], $f : [0, T] \times B[0, MT] \to X$ be measurable in t and Lipschitzean in x, such that $f(t, x) \in F(x)$ for all t, x and $x'(t) = f(t, x(t))$ almost everywhere. By Scorza Dragoni's theorem, we can choose (by possibly modifying E) a compact set $E' \supset E$ such that $m(E') \geq T - \varepsilon/(12M)$ and $f_{|E' \times B[0, MT]}$ is continuous. Let $t_o \in E$ be a density point of E. By the continuity of F we can assume that there exist $\varepsilon > r > 0$ and $1 > \eta > 0$ such that

$$f(t_o, x(t_o)) + rB \subset F(y) \qquad \text{if} \quad |y - x(t_o)| < \eta . \tag{5.7}$$

Let $\sigma > 0$ be such that

$$|f(t, x) - f(s, y)| < \frac{r}{6} \qquad \forall t, s \in E', \ \forall x, y : |t - s| + |x - y| < 2\sigma \tag{5.8}$$

$$H(\text{bd} F(x), \text{bd} F(y)) < \frac{\varepsilon}{6} \qquad \forall x, y : |x - y| < \sigma . \tag{5.9}$$

Let moreover $\sigma \geq \vartheta > 0$ be such that, if $\tau \in [0, T]$ and $|z - x(\tau)| < \vartheta$, then the functions $y : [0, T] \to X$ for which $|y'(t) - f(t, y(t))| < r$ and $y(\tau) = z$ satisfy

$$|y(t) - x(t)| < \sigma \qquad \forall t \in [\tau, T] . \tag{5.10}$$

Let $t_1 > t_o$ be such that

$$t_1 - t_o < \frac{\eta}{M} \wedge \frac{\sigma}{M} \wedge \frac{\vartheta}{2r} \wedge \frac{\varepsilon}{6r} , \tag{5.11}$$

$$m(E \cap [t_o - t_1]) \geq (t_1 - t_o)(1 - \frac{r}{12M}) . \tag{5.12}$$

Set

$$G(t, y) = \begin{cases} f(t, y) & \text{if } t \notin [t_o, t_1] \\ f(t_o, x(t_o)) + rB & \text{if } t \in [t_o, t_1] . \end{cases}$$

Then $G(\cdot, y)$ is measurable, $G(t, \cdot)$ is Lipschitzean and the solutions y of $y' \in G(t, y)$, $y(0) = 0$, by (5.7) and (5.11) are also solutions of (5.1). Moreover, by the same argument as in the proof of Theorem 5.1, the reachable set $R_G(T)$ contains $\overline{\xi}$ in its interior.

We wish to estimate $\mathcal{L}_\theta(y) - L_\theta(\overline{\xi})$ for an arbitrary solution y of $y' \in G(t, y)$, $y(0) = 0$. To this aim, we consider first the distance $\|y - x\|$. If

$t \leq t_o$, then $y(t) = x(t)$; moreover, for every $t \in [t_o, t_1]$ we have

$$
\begin{aligned}
|y(t) - x(t)| &\leq \int_{t_o}^{t_1} |f(t_o, x(t_o)) - f(t, x(t))| \, dt + r(t_1 - t_o) \\
&\leq \int_{[t_o, t_1] \cap E} |f(t_o, x(t_o)) - f(t, x(t))| \, dt \\
&\quad + \int_{[t_o, t_1] \setminus E} |f(t_o, x(t_o)) - f(t, x(t))| \, dt + r(t_1 - t_o) \\
&\leq \frac{r(t_1 - t_o)}{6} + 2M \frac{r(t_1 - t_o)}{12M} + r(t_1 - t_o) < \vartheta \leq \sigma \, ,
\end{aligned}
$$

by (5.8), (5.12) and (5.11). Hence, by (5.10),

$$
|y(t) - x(t)| < \sigma \qquad \forall t \in [t_o, T] \, .
$$

This estimate, together with (5.8), (5.9), (5.11) and (5.12), implies

$$
\begin{aligned}
\mathcal{L}_\vartheta(y) - \mathcal{L}_\vartheta(x) &= \int_{t_o}^{T} \left(d(y'(t), \mathrm{bd} F(y(t))) - d(f(t, x(t)), \mathrm{bd} F(x(t))) \right) dt \\
&\geq - \int_{t_o}^{T} |y'(t) - f(t, x(t))| \, dt - \int_{t_o}^{T} H(\mathrm{bd} F(y(t)), \mathrm{bd} F(x(t))) \, dt \\
&\geq - \int_{[t_o, t_1] \cap E'} |f(t_o, x(t_o)) - f(t, x(t))| \, dt \\
&\quad - \int_{[t_1, T] \cap E'} |f(t, y(t)) - f(t, x(t))| \, dt - 2M \mathrm{m}([t_o, T] \setminus E') \\
&\quad - r(t_1 - t_o) - \int_{t_o}^{T} H(\mathrm{bd} F(y(t)), \mathrm{bd} F(x(t))) \, dt \\
&\geq - \left(\frac{r}{6} + \frac{r}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \right) \\
&> -\varepsilon \, .
\end{aligned}
$$

Since for every ξ in a neighborhood of $\overline{\xi}$ there exists a solution y of (5.1) reaching it such that $\mathcal{L}_\vartheta(y) - L_\vartheta(\overline{\xi}) = \mathcal{L}_\vartheta(y) - \mathcal{L}_\vartheta(x) \geq -\varepsilon$, we have that $L_\vartheta(\xi) - L_\vartheta(\overline{\xi}) \geq -\varepsilon$, and the proof is concluded. $\triangle$

Remark 5.3. If $\mathrm{int} F(x) \neq \emptyset$ for all x, then the preceding results remain true also for the likelihood $\mathcal{L}_b$ of Definition 4.2.

6. APPENDIX: ON CONVEX BODY VALUED MULTIFUNCTIONS

Let F be a Hausdorff continuous multifunction from a metric space X into the closed, convex, bounded subsets of a normed space Y with nonempty interior. Some continuity properties of such maps, as well as the existence of continuous selections, were established in [15] and used by the same authors in the analysis of infinite dimensional differential inclusions [14]. In particular, in [15], Lemma 3.1 and Proposition 3.7, it was proved that the map

$$x \mapsto \rho(x) := \sup\{r > 0 \mid \exists y \in F(x) : B[y,r] \subset F(x)\}$$

is continuous, and the multifunction

$$x \mapsto F_{\mu(x)}(x) := \{y \in F(x) \mid d(y, \mathrm{bd}F(x)) \geq \mu(x)\}$$

is Hausdorff continuous, where $0 < \mu(x) < \rho(x)/2$ is a given continuous function.

Aim of this paragraph is to further investigate the maps $\rho(\cdot)$ and $F_{\mu(\cdot)}(\cdot)$, with $\mu(\cdot) = \rho(\cdot)$. Its main results assert that $x \mapsto F_{\rho(x)}(x)$ is upper semicontinuous (in finite dimension) and may not be lower semicontinuous; moreover, if F is Lipschitzean and F_ρ is single-valued (as it is the case when the values of F are strictly convex bodies in $\mathbb{R}^n$), the continuous selection $x \mapsto F_{\rho(x)}(x)$ may fail to be Lipschitzean. Therefore the point $F_{\rho(x)}(x)$, i.e. the center of the ball inscribed in $F(x)$, enjoys a feature similar to the Čebyšev center, i.e. the center of the ball circumscribed to $F(x)$ (see [5], p.75). We state most of the results in infinite dimensions, for the sake of completeness.

According to the previous Definition 4.1, the solutions of $x' \in F_{\rho(x)}(x)$, $x(0) = 0$ are those in S_F with maximal ∂-likelihood in the myopic sense. The Example 6.2 below shows that there can be solutions of (5.1) which have maximal myopic ∂-likelihood but not intertemporal maximal ∂-likelihood.

In what follows, we denote by $\mathcal{B}$ the set of all bodies in Y, i.e. the closed, convex, bounded subsets of Y with nonempty interior.

Let X be a metric space and Y a reflexive Banach space, with open unit ball B. For any $A \in \mathcal{B}$ we define

$$\rho_A := \sup\{r > 0 \mid \exists y \in A : y + rB \subset A\}, \tag{6.1}$$

and

$$C_A := \{y \in A \mid y + \rho_A B \subset A\}. \tag{6.2}$$

In other words, C_A is the set of all centers of the balls inscribed in A.

The first result concerns C_A.

Proposition 6.1. *For any $A \in \mathcal{B}$, the set C_A is nonempty, convex and closed. If Y is finite dimensional and A is strictly convex, then C_A is a singleton.*

Proof. Let $(y_n)_{n \geq 1}$ be a sequence in A such that

$$y_n + (\rho_A - 1/n)B \subset A.$$

By the weak compactness of A, we can suppose that $y_n \rightharpoonup y \in A$. Let w be in $y + \rho_A B$. By Mazur's theorem there exists a sequence of convex combinations $z_n = \sum_{i=0}^{k_n} \lambda_i^n y_{n+i}$, with $z_n \to y$ strongly. Hence, if n is sufficiently large, $\|w - z_n\| < \rho_A - 1/n$ and therefore $w \in A$, because $w = w - z_n + z_n = \sum_{i=0}^{k_n} \lambda_i^n (w - z_n + y_{n+i})$ is a convex combination of points in A. Hence C_A is nonempty.

The convexity of C_A is straightforward, while the closure follows from

$$C_A = \{y \in A \mid d(y, \mathrm{bd}A) = \rho_A\} \tag{6.3}$$

(see [15], Lemma 3.2).

Let now Y be finite dimensional and A strictly convex, and assume by contradiction that $y_1, y_2 \in C_A$, $y_1 \neq y_2$. Set z to be $(y_1 + y_2)/2$ and notice that $d(z, \mathrm{bd}A) = \rho_A$. Choose $w \in \mathrm{bd}A$ such that $\|w - z\| = \rho_A$. Let $\mathcal{H}$ be the hyperplane orthogonal to $w - z$: it is a support hyperplane for A and therefore $(w + \mathcal{H}) \cap A$ is contained in $\mathrm{bd}A$. We have also that $y_2 - y_1$ is orthogonal to $w - z$, otherwise $d(y_1, \mathrm{bd}A) < \rho_A$ or $d(y_2, \mathrm{bd}A) < \rho_A$. Hence $y_i + w - z \in w + \mathcal{H}$, $i = 1, 2$, and therefore $\mathrm{bd}A$ contains a segment, a contradiction.
$\triangle$

Let $F : X \to \mathcal{B}$ be a multifunction. We set $\rho_F(x) := \rho_{F(x)}$ and $C_F(x) := C_{F(x)}$. The following results hold.

Proposition 6.2. *Let $F : X \to \mathcal{B}$ be a Hausdorff continuous multifunction. Then the map $x \mapsto C_F(x)$ is upper semicontinuous, provided Y is endowed with the weak topology.*

Before the proof, we state a technical lemma.

Lemma 6.1. *Let Y be a normed space and $K \subset Y$ a convex set. Then the map*

$$d_K : K \to d(y, \mathrm{bd}K)$$

is (continuous and) concave.

Proof. Let y, z be in K. Since K is convex, the set

$$H = \mathrm{co}\{B[y, d(y, \mathrm{bd}K)] \cup B[z, d(z, \mathrm{bd}K)]\}$$

is contained in K. Fix $\lambda \in (0,1)$. Then

$$d(\lambda y + (1-\lambda)z, \mathrm{bd}K) \geq d(\lambda y + (1-\lambda)z, \mathrm{bd}H)$$
$$\geq \lambda d(y, \mathrm{bd}K) + (1-\lambda)d(z, \mathrm{bd}K) \ .$$

$\triangle$

Proof of Proposition 6.2. Let $\tau(X)$ and $\sigma(Y)$ be, respectively, the metric topology in X and the weak topology in Y. It suffices to prove that the graph of $C_F(\cdot)$ ($\mathrm{graph}(C_F) = \{(x,y) \in X \times Y \mid y \in F(X)\}$) is closed in $\tau(X) \times \sigma(Y)$. To this aim, we recall first that by [15], Proposition 2.1, Lemma 3.1, the maps $x \mapsto Y \setminus F(x)$ and $x \mapsto \mathrm{bd}F(x)$ are Hausdorff continuous and the map $x \mapsto \rho_F(x)$ is continuous. Let $(x_n)_{n\geq1}$, $(y_n)_{n\geq1}$ be sequences such that $y_n \in C_F(x_n)$, $x_n \to x$ in X and $y_n \rightharpoonup y$ weakly in Y. We want to show that $y \in C_F(x)$. We remark first that, for n large enough, $y_n \in F(x)$, because $d(y_n, Y \setminus F(x)) \geq d(y_n, Y \setminus F(x_n)) - H(Y \setminus F(x_n), Y \setminus F(x)) > 0$. Then, let by Mazur's theorem $z_n = \sum_{i=0}^{k_n} \lambda_i^n y_{n+i}$ be a sequence strongly converging to y and fix $\varepsilon > 0$. By Lemma 6.1 and the continuity of $\mathrm{bd}F(\cdot)$ and $\rho_F(\cdot)$, for n large enough we have

$$d(z_n, \mathrm{bd}F(x)) = d(\sum_{i=0}^{k_n} \lambda_i^n y_{n+i}, \mathrm{bd}F(x))$$

$$\geq \sum_{i=0}^{k_n} \lambda_i^n d(y_{n+i}, \mathrm{bd}F(x))$$

$$\geq \sum_{i=0}^{k_n} \lambda_i^n \left(d(y_{n+i}, \mathrm{bd}F(x_{n+i})) - H(\mathrm{bd}F(x), \mathrm{bd}F(x_{n+i}))\right)$$

$$\geq \rho_F(x) - \varepsilon \, ,$$

where the last inequality is obtained recalling (6.3). Since ε is arbitrary, we have that $d(y, \mathrm{bd}F(x)) \geq \rho_F(x)$, concluding the proof.

$\triangle$

Corollary 6.1. *Let Y be finite dimensional and $F(x)$ be strictly convex for every $x \in X$. Then the (single-valued) map $x \mapsto C_F(x)$ is continuous.*

Example 6.1. The map $C_F(x)$ need not be lower semicontinuous.

In fact, it suffices to let $F : [0,1] \to 2^{\mathbb{R}^2}$ be defined as

$$F(x) = \mathrm{co}\{B \cup ((3,0) + (x+1)B)\} \, .$$

Then $C_F(0)$ is the segment $[(0,0),(3,0)]$, while for any $x > 0$ $C_F(x)$ is the point $(3,0)$.

$\triangle$

We consider, finally, the case where F is Lipschitz continuous. We show that, although the radius $\rho_F(\cdot)$ of the inscribed ball has the same Lipschitz constant as $F(\cdot)$, its center (when it is unique) may fail to be Lipschitzean, even in $\mathbb{R}^2$.

Proposition 6.3. *Let $F : X \to B$ be Lipschitzean with Lipschitz constant L. Then the map $\rho_F : X \to \mathbb{R}^+$ is Lipschitzean with the same constant L.*
Proof. The same argument of Lemma 3.1 in [15] can be applied also to obtain the Lipschitz continuity. Indeed, assume by contradiction that there exist $x_1, x_2 \in X$ such that $|\rho_F(x_1) - \rho_F(x_2)| = Ld(x_1, x_2) + \eta$, $\eta > 0$. If $\rho_F(x_2) > \rho_F(x_1)$, take $y \in F(x_2)$ such that

$$B[y, \rho_F(x_1) + Ld(x_1, x_2) + \eta/2] \subset F(x_2).$$

By the Lipschitz continuity, we then have

$$B[y, \rho_F(x_1) + Ld(x_1, x_2) + \eta/2] \subset F(x_1) + Ld(x_1, x_2)B,$$

which by the convexity of $F(x_1)$ implies $B[y, \rho_F(x_1) + \eta/2] \subset F(x_1)$, a contradiction. If $\rho_F(x_1) > \rho_F(x_2)$ a symmetric argument can be used.

$$\triangle$$

Example 6.2. a) A Lipschitzean map $F : [0,1] \to 2^{\mathbb{R}^2}$, with compact and strictly convex values with nonempty interior, such that $t \mapsto C_F(t)$ is not Lipschitzean.
   Set

$$F(0) = \{x^2 + y^2 \le 1\} \cup \{\frac{x^2}{4} + y^2 \le 1, \ x \ge 0\}$$

and, for $0 < t \le 1$,

$$F(t) = \{(x - \sqrt{t})^2 + y^2 \le 1 - \frac{t}{3}\}$$
$$\cup \{\frac{x^2}{4} + y^2 \le 1, \ x \ge \frac{4}{3}\sqrt{t}\}$$
$$\cup \{a(t)x^2 + b(t)y^2 + c(t)x \le 1, \ -\sqrt{\frac{t}{2}} \le x \le \sqrt{t}\}$$
$$\cup \{x^2 + y^2 \le 1, \ -1 \le x \le \frac{-\sqrt{t}}{2}\},$$

where

$$a(t) = \frac{-\sqrt{t}}{(4\sqrt{2} + 4)t - 9\sqrt{2} - 12},$$

$$b(t) = \frac{-9\sqrt{12} - 12}{(4\sqrt{2} + 4)t - 9\sqrt{2} - 12},$$

$$c(t) = \frac{2\sqrt{2}\sqrt{t}}{(4\sqrt{2} + 4)t - 9\sqrt{2} - 12}.$$

The set $F(0)$ consists of the unit ball B of $I\!\!R^2$ plus half of the ellipse A through $(2,0)$, tangent to B at $(0,\pm1)$. The set $F(t)$, contained in $F(0)$, is made of the ball C centered at $(\sqrt{t},0)$ and tangent to A at $(\frac{4}{3}\sqrt{3}, \pm\sqrt{1 - \frac{t}{3}})$, plus a subset of A and a subset contained in B of the ellipse through $(-\sqrt{\frac{t}{2}}, \pm\sqrt{1 - \frac{t}{2}})$, $(\sqrt{t}, \pm\sqrt{1 - \frac{t}{3}})$. Notice also that, if $0 \le t_1 \le t_2 \le 1$, then $F(t_2) \subset F(t_1)$.

The Hausdorff distance between $F(t_1)$ and $F(t_2)$ is less than the Hausdorff distance between the two curves

$$\left\{ a(t_1)x^2 + b(t_1)y^2 + c(t_1)x = 1, \ -\sqrt{\frac{t_1}{2}} \le x \le \sqrt{t_1} \right\}$$

and

$$\left\{ a(t_2)x^2 + b(t_2)y^2 + c(t_2)x = 1, \ -\sqrt{\frac{t_2}{2}} \le x \le \sqrt{t_2} \right\},$$

which is less than $15|t_1 - t_2|$. Therefore the map $F(t)$ is Lipschitzean. On the other hand, the circle inscribed in $F(t)$ is $\{(x - \sqrt{t})^2 + y^2 \le 1 - \frac{1}{3}\}$, whose center $(\sqrt{t},0)$ is not Lipschitzean.

b) Let B be the unit ball in $I\!\!R^2$. Set $G : B \to 2^{I\!\!R^2}$ to be

$$G(x) = F(\|x\|),$$

where F is as in Part a). Then G is Lipschitzean and $C_G(x) = (\sqrt{\|x\|}, 0)$. The solution of $x' \in G(x)$, $x(0) = 0$ with maximal $\mathcal{L}_{\vartheta}$-likelihood (equal to 1) is the constant 0, while $x' = C_G(x)$, $x(0) = 0$ admits infinitely many solutions x with $\mathcal{L}_{\vartheta}(x) < 1$.

$\triangle$

REFERENCES

1. P. Antonelli and V. Křivan, Fuzzy differential inclusions as substitutes for stochastic differential equations in population biology, 1991, to appear.

2. Z. Artstein, Extensions of Lipschitz selections and an application to differential inclusions, preprint, The Weizmann Institute of Science, Rehovot, Israel.

3. J.-P. Aubin, A survey of viability theory, *SIAM J. Control & Optimization* **28** (1990), 749-788.

4. J.-P. Aubin, *Viability Theory*, preprint.

5. J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.

6. J.-P. Aubin and H. Frankowska, Heavy viable trajectories of controlled systems, *Ann. Inst. H. Poincaré, Analyse Nonlinéaire* **2** (1985), 371-395.

7. J.-P. Aubin and H. Frankowska, *Set-valued Analysis*, Birkhäuser, Boston, 1990.

8. A. Bressan, The most likely path of a differential inclusion, *J. Diff. Equations* **88** (1990), 155-174.

9. A. Bressan and G. Colombo, Generalized Baire category and differential inclusions in Banach spaces, *J. Diff. Equations* **76** (1988), 135-158.

10. A. Cellina and R. M. Colombo, Some qualitative and quantitative results on a differential inclusion, *Rend. Sem. Mat. Univ. e Politecnico di Torino*, to appear.

11. A. Cellina and C. Mariconda, Kuratowski's index of a decomposable set, *Bulletin of the Polish Academy of Sci.* **37** (1989), 679-685.

12. L. Cesari, *Optimization-Theory and Applications*, Springer, New York, 1983.

13. R.M. Colombo and V. Křivan, Selective strategies in food webs, S.I.S.S.A., preprint (1991).

14. F.S. De Blasi and G. Pianigiani, A Baire category approach to the existence of solutions to multivalued differential inclusions, *Funkcial. Ekvac.* **25** (1982), 153-162.

15. F.S. De Blasi and G. Pianigiani, Remarks on Hausdorff continuous multifunctions and selections, *Comm. Math. Univ. Carol.* **24** (1983), 553-561.

16. M. Falcone and P. Saint-Pierre, Slow and quasi-slow solutions of differential inclusions, *Nonlinear Analysis* **11** (1987) 367-377.

17. A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer, Dordrecht (1988).

18. N. Kikuchi, On contingent equations satisfying the Carathéodory type conditions, *Publ. R.I.M.S., Kyoto Univ. Ser. A* **3** (1968), 361-371.

19. S. Lojasiewicz jr., Parametrization of convex sets, *J. Approximation Theory* (1991), to appear.

20. S. Lojasiewicz jr., A. Pliś and R. Suarez, Necessary conditions for a nonlinear control system, *J. Diff. Equations* **59** (1985), 257-265.

21. A. Ornelas, Parametrization of Carathéodory multifunctions, *Rend. Sem. Mat. Univ. Padova* **83** (1990), 33–44.